Topic - Partial Differentiation   Date -

class . B.Sc I (Hons, Sub)

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Study Material :- Partial differentiation

Let $u = f(x, y)$

then $\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

is called partial differential co-efficient of $f(x, y)$ with respect to x provided limit exists.

Similarly

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

is called partial differential co-efficient of $u = f(x, y)$ with respect to x provided limit exists.

other Notations →

$\frac{\partial u}{\partial x}$ or u_x or $\frac{\partial f}{\partial x}$ or f_x

is partial derivative of $u = f(x, y)$ with respect to x

$\frac{\partial u}{\partial y}$ or u_y or $\frac{\partial f}{\partial y}$ or f_y is partial derivative of $u = f(x, y)$ with respect to y.

$f_x \Rightarrow$ we have to differentiate $u = f(x, y)$ with respect to x treating y as constant

$f_y \Rightarrow$ we have to differentiate $u = f(x, y)$ with respect to y treating x as const.

Euler's theorem on Partial differentiation of homogeneous function of two independent variables

Statement :→

If $f(x,y)$ be a homogeneous function of degree n

then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

Proof :→ let

$$f(x,y) = Ax^{\alpha_1}y^{\beta_1} + Bx^{\alpha_2}y^{\beta_2} + Cx^{\alpha_3}y^{\beta_3} + \cdots \quad (1)$$

Where $\alpha_1+\beta_1 = \alpha_2+\beta_2 = \cdots = \alpha_n+\beta_n = n$

Here $A, B, C \cdots$ are constant

i.e independent of x and y.

Diff (1) partially with respect to x

$$\frac{\partial f}{\partial x} = \alpha_1 A x^{\alpha_1 - 1}y^{\beta_1} + \alpha_2 B x^{\alpha_2-1}y^{\beta_2} + C\alpha_3 x^{\alpha_3-1}y^{\beta_3} + \cdots$$

$$\therefore x\frac{\partial f}{\partial x} = \alpha_1 A x^{\alpha_1}y^{\beta_1} + \alpha_2 B x^{\alpha_2}y^{\beta_2} + \alpha_3 C x^{\alpha_3}y^{\beta_3} + \cdots$$

Again Diff. (1) Partially with respect to y

$$\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1-1} + B\beta_2 x^{\alpha_2}y^{\beta_2-1} + C\beta_3 x^{\alpha_3}y^{\beta_3-1} + \cdots$$

$$\Rightarrow y\frac{\partial f}{\partial y} = A\beta_1 x^{\alpha_1}y^{\beta_1} + B\beta_2 x^{\alpha_2}y^{\beta_2} + C\beta_3 x^{\alpha_3}y^{\beta_3} + \cdots$$

Now $x\frac{\partial f}{\partial x} + y\frac{\partial y}{\partial y} = Ax^{\alpha_1}y^{\beta_1}(\alpha_1+\beta_1) + Bx^{\alpha_2}y^{\beta_2}(\alpha_2+\beta_2) + \cdots$

$= Ax^{\alpha_1}y^{\beta_1}\, n + Bx^{\alpha_2}y^{\beta_2}\, n + \cdots$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$$

Similarly, if $u = f(x,y,z)$ be a homogeneous function of three independent variables x, y, z of degree n

then $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n f(x,y,z)$$

2. Show that -

If u be a homogeneous function of two variables x and y of degree n, then prove that -

(i) $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$$

(ii) $$\cancel{x\frac{\partial^2 u}{\partial x^2}}\ x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

(iii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Proof → Here u is homogeneous function of two independent variables x and y

then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{—— (1)}$$

Diff. (i) partially with respect to x.

$$\frac{\partial u}{\partial x} + x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) = n\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial y}{\partial x}\cdot\frac{\partial u}{\partial y} + y\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = n\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + 0 + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= (n-1)\frac{\partial u}{\partial x}$$

Similarly we may show

(ii) $$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

(iii) Since

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)x\frac{\partial u}{\partial x} \quad \text{— (A)}$$

Also $$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \quad \text{— A}$$

$$\Rightarrow xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)y\frac{\partial u}{\partial y} \quad \text{— (B)}$$

Adding (A) and (B)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= (n-1)\, n\, u$$

Hence